M-math (2013) 1st year Mid Semester Exam
Subject : Advanced Probability

Time : 2.30 hours Max.Marks 50.

1. Let $X_1, \cdots, X_n, \cdots$ be an i.i.d sequence with $X_i \sim N(0,1), i = 1, \cdots$. Let $S_n = X_1 + \cdots + X_n$. Compute the limit

$$\lim_{n\to\infty} \frac{1}{n} \log \mathrm{P}\{\mathrm{S_n} \geq \mathrm{a_n x}\}$$

for $x > 0$ and for the two cases : a) $a_n = n$ and b) $a_n = n^{\frac{3}{2}}$. Describe the rate function for the large deviation principle in these two cases. (5+5)

2. Let ν be a measure on the Borel sigma field of $(0, \infty)$ such that $\int_0^\infty (1 \wedge x)\nu(dx) < \infty$. Let α be a real number. Show that there exists a non negative, infinitely divisible random variable X whose Laplace transform is given by

$$E(e^{-tX}) = e^{[\alpha t + \int_0^\infty (1-e^{-tx})\nu(dx)]}$$

for $t \geq 0$. (10)

3. Let ν be a measure on the Borel sigma field of $\mathbb{R} \setminus \{0\}$ having density $\frac{C}{|x|^2}$ with respect to Lebesgue measure and $C > 0$ is a constant.
a) Verify that ν is a Levy measure on $\mathbb{R} \setminus \{0\}$.
b) Suppose that X is an infinitely divisible random variable with Levy-Khinchine representation $E(e^{itX}) = e^{\psi(t)}$ where

$$\psi(t) = \int\limits_{\mathbb{R}\setminus\{0\}} (e^{itx} - 1 - itxI_{\{|x|\leq 1\}})\nu(dx),$$

where $i = \sqrt{-1}$. Show that we can choose the constant $C > 0$ so that $\psi(t) = -|t|$ and consequently that X has the Cauchy distribution.(Hint: Use the fact that the measure ν is symmetric to evaluate the integral.)
c) Show that X has the stable distribution with parameter $\alpha = 1$ i.e. for every $n \geq 1$, there exists i.i.d. random variables $X_1, \cdots, X_n$, with each X_i having the same distribution as X, and suitable constants a_n and b_n such that $X_1 + \cdots + X_n$ has the same distribution as $a_nX + b_n$. (5+10+5)

4. Let $\{X_i; 1 \leq i \leq n\}$ be integrable , i.i.d. random variables. Let $S_n :=$ $X_1 + \cdots + X_n$. Show that $E[X_i|S_n] = \frac{S_n}{n}, i = 1, \cdots, n.$ (10)

5. Let $X_1, X_2,$ be independent random variables having an exponential distribution with a parameter $\theta > 0$. Compute $E(X_1 \wedge X_2)|X_1)$ explicitly as a function of X_1 . (10)